Properties of Circles

From the Canadian Arctic to Southern Africa, people have developed ingenious circular structures to use as homes or temporary shelters. The materials available and the climate greatly influence the design of these structures.

Investigate

What properties do circles have?

Method 1: Use Pencil and Paper

1. Draw a circle on a sheet of paper by tracing around a circular object such as a juice can.

2. Find the centre of the circle by folding your drawing. Explain how you know where the centre is.

3. Reflect What property do diameters of a circle have?

4. Mark any two points on the circle and label them A and B. Join the points to form a chord of the circle. Draw the right bisector of this chord. What property does this right bisector have? Check whether the right bisectors drawn by your classmates have the same property.

5. Choose any point on a blank sheet of paper, but do not mark this point. Instead, mark three points that are all about the same distance from the first point. Label these three points P, Q, and R. Exchange your set of points with a classmate.

6. Find the centre of the circle that passes through the three points that your classmate marked. Explain your method. Draw the circle that passes through the points.

7. Reflect What property do the chords of a circle have?
Method 2: Use *The Geometer’s Sketchpad*®

1. Turn on the grid display. Construct a point at the centre of the grid and a second point near the edge. Using the first point as the centre, construct a circle by choosing Circle by Centre and Point from the Construct menu.

2. Construct any two points on the circle. Construct a chord by joining these two points with a line segment.

3. Construct the right bisector of the chord.

4. Select one of the endpoints of the chord. From the Display menu, choose Animate Point. As the endpoint of the chord moves around the circle, watch the right bisector of the chord. What property does this bisector have?

5. Reflect What property do the chords of a circle have? How could you use this property to find the centre of a circle given three points on the circumference?

Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. Check that the axes are displayed. Choose Circle from the F2 menu. Place the centre near the middle of the screen, and place the end of the radius close to the top of the screen.

2. Choose Segment from the F2 menu. Construct a chord by using a line segment to join any two points on the circle.

3. Choose Perp. Bis. from the F3 menu, and construct the right bisector of the chord.

4. Move the cursor to one endpoint of the chord, and press ALPHA. Watch the right bisector of the chord as you move the endpoint around the circle. What property does this bisector have?

5. Reflect What property do the chords of a circle have? How could you use this property to find the centre of a circle given three points on the circumference?
Example 1 Right Bisector of a Chord

Verify that the centre of this circle lies on the right bisector of the chord AB.

Solution

The centre lies on the right bisector of AB only if the coordinates (0, 0) satisfy the equation of the right bisector.

The midpoint of AB is on the right bisector. Use the coordinates of A and B to find the midpoint and the slope of AB. Then, calculate the slope of a line perpendicular to AB. Use this slope and the coordinates of the midpoint, M, to determine the equation of the right bisector of AB.

Find the midpoint coordinates and the slope of AB.

\[
M(x, y) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
= \left( \frac{2 + 4}{2}, \frac{-4 + 2}{2} \right) \quad = \frac{2 - (-4)}{4 - 2}
\]

\[
= (3, -1) \quad = 3
\]

Since the right bisector is perpendicular to AB, the slope of this bisector is the negative reciprocal of \(m_{AB}\).

\[
- \frac{1}{m_{AB}} = - \frac{1}{3}
\]

Use this slope and the coordinates M(3, -1) to find the y-intercept.

\[
y = mx + b
\]

\[
-1 = - \frac{1}{3} (3) + b
\]

\[
-1 = -1 + b
\]

\[
0 = b
\]

The equation of the right bisector is \(y = - \frac{1}{3} x\). The coordinates (0, 0) satisfy this equation. Therefore, the centre of the circle lies on the right bisector of the chord PQ.
Example 2  Points on a Circle

a) Show that the points P(9, -3), Q(8, 6), and R(-1, 5) lie on a circle with its centre at C(4, 1).

b) Does any other circle pass through points P, Q, and R? Explain.

Solution

a) If the three points lie on a circle centred at (4, 1), each point must be the same distance from (4, 1). Compare the lengths of CP, CQ, and CR.

\[
CP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(9 - 4)^2 + (-3 - 1)^2} \\
= \sqrt{5^2 + (-4)^2} \\
= \sqrt{41}
\]

\[
CQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(8 - 4)^2 + (6 - 1)^2} \\
= \sqrt{4^2 + 5^2} \\
= \sqrt{41}
\]

\[
CR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{(-1 - 4)^2 + (5 - 1)^2} \\
= \sqrt{(-5)^2 + 4^2} \\
= \sqrt{41}
\]

Since CP = CQ = CR, the points P, Q, and R all lie on a circle centred at C(4, 1).

b) The right bisector of PQ includes all points that are equidistant from P and Q. Similarly, the right bisector of QR includes all points that are equidistant from Q and R. These two lines meet only at point C(4, 1). There is no other point equidistant from P, Q, and R.
Therefore, the circle with centre $C(4, 1)$ and radius $\sqrt{41}$ is the only circle that passes through the points $P$, $Q$, and $R$.

You cannot determine the centre of a circle from just two points on the circumference. The centre can lie anywhere on the right bisector of the chord between the two points.

**Key Concepts**

- The diameters of a circle intersect at the centre of the circle.

- The right bisector of a chord of a circle passes through the centre of the circle.

- The right bisectors of two chords of a circle intersect at the centre of the circle.

- There is only one circle that passes through three given non-collinear points.

**Communicate Your Understanding**

**C1** Describe how to find the point that is equidistant from three given points by

- a) folding a plot of the given points
- b) constructing lines
- c) using analytic geometry

**C2** Describe how you would determine the balance point for a flat circular object.
Practise

For help with question 1, see Example 1.

1. a) Find the coordinates of the midpoint, M, of AB.
   b) Find the slope of the chord AB.
   c) Verify that OM is perpendicular to AB.

   ![Graph showing midpoint M and slopes]

   For help with questions 2 to 4, see Example 2.

2. a) Verify that the points \(P(-1, -2), Q(2, 7),\) and \(R(6, 5)\) are equidistant from the point \(C(2, 2)\).
   b) Draw the circle that passes through points P, Q, and R.

3. a) Verify that the points \(A(12, 6), B(4, 10),\) and \(C(0, 2)\) lie on a circle with its centre at \(D(6, 4)\).
   b) Determine the length of the radius of the circle.
   c) Plot points A, B, and C on grid paper, and draw the circle that passes through the points. Use your drawing to check your answers to parts a) and b).

4. a) Verify that the points \(E(-5, 0), F(-2, 3),\) and \(G(6, -11)\) lie on a circle with its centre at \(H(2, -4)\).
   b) Determine the length of the radius of the circle.

   ![Graph showing points and circle]

5. Verify that the centre of this circle lies on the right bisector of the chord PQ.

   ![Circle with points P, Q, and centre]

6. a) Explain how you know that the origin is the centre of the circle defined by the equation \(x^2 + y^2 = 45\).
   b) Verify that the points \(R(-3, 6)\) and \(S(-6, -3)\) lie on the circle.
   c) Verify that the line through the origin and the midpoint of the chord RS is perpendicular to the chord.

7. A machinist needs to drill a hole in the centre of a circular part. Describe how the machinist could mark the correct location for this hole.

8. a) You have 3.0 m of edging to put around a flower bed. Find the maximum area you can enclose if the shape of the flower bed is an equilateral triangle.
   b) Find the maximum area you can enclose if the flower bed is square.
   c) Find the maximum area you can enclose if the flower bed is circular.
   d) What property of circles makes them a useful shape for the base of storage tanks and some types of buildings?

9. Find the centre of the circle that passes through the points \(A(-7, 4), B(-4, 5),\) and \(C(0, 3)\).
10. **Use Technology** Use geometry software to answer question 9. Outline your method.

11. Three friends live in Sudbury, Toronto, and Windsor. They are planning to go camping together and want to find a park that is approximately the same distance from each of their homes. Describe how the friends could fold an Ontario roadmap to help them find a suitable campground.

12. On a town map, the coordinates of three schools are J(8, 13), K(10, 7), and L(14, 15). The town is planning to build a new swimming pool that is the same distance from all three schools. Determine the coordinates for the pool.

13. Draw a circle with centre O. Add any chord PQ, with midpoint M. What can you conclude about \( \triangle OMP \) and \( \triangle OMQ \)? Explain your reasoning.

14. Draw any circle. Draw a diameter of the circle and label its endpoints J and K. Let L be any other point on the circumference of the circle. Use angle sums in triangles to show that \( \triangle JKL \) is a right triangle.

15. **Use Technology** Use geometry software to answer question 14. Outline your method.

**Extend**

16. To find a good location for a community hospital, planners could find the smallest circle that encloses all the homes on a map of the community.
   
a) What is the advantage of a location at the centre of the smallest enclosing circle?

b) What other factors might prevent the centre from being the best location for the hospital?

c) Using grid paper or geometry software, plot 15 points to represent neighbourhoods. Try to find the smallest circle that encloses these points. Describe the method that you used. Mark where you would place the hospital for the neighbourhoods that the dots represent. Explain why you chose this location.

17. A pilot filed a flight plan that listed a cruising speed of 160 km/h with enough fuel on board for 3.5 h of flying. After 2 h, the aircraft passed over Lake Traverse. The aircraft was reported missing when it failed to reach its planned destination.
   
a) Sketch a diagram showing the area where the plane may have crashed.

b) How large is this area?

c) Which part of the search area should be searched first? Explain your reasoning.

18. **Math Contest** Show how \( \angle ABC \) and \( \angle AOC \) in quadrilateral ABCD are related.

19. **Math Contest** Keshawn and Samantha are in a science class with 18 other students. The teacher randomly divides the class into 10 pairs of laboratory partners. The probability that Keshawn and Samantha are laboratory partners is

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