A water garden combines a pond with aquatic plants and often ornamental fish, such as koi, to add visual appeal to the landscape. The area of a rectangular water garden can be represented by the quadratic expression \( x^2 + 5x + 6 \). To find the length and the width of the rectangle, you can write the trinomial as the product of two binomials.

**Investigate A**

**How can you use a model to factor quadratic expressions of the form \( x^2 + bx + c \)?**

1. To factor \( x^2 + 5x + 6 \), use algebra tiles to create a rectangular area whose length and width represent the factors of the trinomial.
   a) Arrange one \( x^2 \)-tile, five \( x \)-tiles, and six unit tiles to form a rectangle with area \( x^2 + 5x + 6 \). Place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.
   b) Write the equation for the trinomial as a product of the binomial dimensions.

2. Repeat step 1 for each trinomial.
   a) \( x^2 + 6x + 5 \)  
   b) \( x^2 + 3x + 2 \)  
   c) \( x^2 + 4x + 3 \)  
   d) \( x^2 + 6x + 8 \)

3. Each trinomial in steps 1 and 2 is of the form \( x^2 + bx + c \). What do you notice about \( b \) and \( c \) and the binomial factors for each trinomial? Describe the relationship.

4. Test your conclusions from step 3 on each trinomial. Use algebra tiles to check your answer.
   a) \( x^2 + 7x + 6 \)  
   b) \( x^2 + 8x + 12 \)

5. **Reflect** Describe a process for finding the factors of a quadratic expression of the form \( x^2 + bx + c \).
Investigate B

How can you use patterns to factor quadratic expressions of the form $x^2 + bx + c$?

A: Positive Values of $b$ and $c$

1. Expand and simplify each product. Try to apply the distributive property mentally.
   a) $(x + 4)(x + 3)$
   b) $(x + 1)(x + 5)$
   c) $(x + 7)(x + 8)$

2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$. Describe how you calculated $b$ and $c$ using the values of $r$ and $s$.

3. Use the patterns from step 2 to reverse the process. Write each trinomial of the form $x^2 + bx + c$ as a binomial product of the form $(x + r)(x + s)$.
   a) $x^2 + 6x + 8$
   b) $x^2 + 7x + 10$
   c) $x^2 + 9x + 20$
   d) $x^2 + 10x + 21$

4. Reflect Describe a process for factoring quadratic expressions of the form $x^2 + bx + c$.

B: Negative Values of $b$ and/or $c$

1. Expand and simplify each product. Try to apply the distributive property mentally.
   a) $(x - 3)(x - 2)$
   b) $(x - 1)(x - 5)$
   c) $(x - 1)(x + 5)$
   d) $(x + 3)(x - 8)$

2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$.
   a) Describe how you determined the signs of the values of $b$ and $c$ when both values of $r$ and $s$ were negative.
   b) Describe how you determined the signs of the values of $b$ and $c$ when only one of the values of $r$ and $s$ was negative.

3. Use your process to factor each trinomial.
   a) $x^2 - 7x + 10$
   b) $x^2 + 4x - 5$
   c) $x^2 - 4x - 5$
   d) $x^2 - 3x - 10$

4. Reflect How does your process for factoring quadratic expressions of the form $x^2 + bx + c$ change when the values of $b$ and/or $c$ are negative?
By finding the dimensions of a rectangle whose area is a quadratic expression, you are reversing the process of expanding two binomials that you learned in Section 5.1. This process is called factoring.

Another way to factor a quadratic expression of the form \( x^2 + bx + c \) is to study the patterns from multiplying two binomials.

\[
(x + r)(x + s) = x^2 + sx + rx + rs = x^2 + (r + s)x + rs
\]

Therefore, \( x^2 + bx + c = (x + r)(x + s) \), where \( r + s = b \) and \( r \times s = c \).

In general, you will factor over the integers, meaning that the values of \( r \) and \( s \) are integers only.

Many quadratic expressions, such as \( x^2 + 3x + 5 \), cannot be factored over the integers. No two integers have a product of 5 and a sum of 3.

**Example 1  Factor Quadratic Expressions**

Factor, if possible.

a) \( x^2 + 7x + 12 \)  
   b) \( x^2 + 4x + 6 \)  
   c) \( x^2 - 29x + 28 \)  
   d) \( x^2 + 3x - 18 \)  
   e) \( x^2 - 4x - 21 \)

**Solution**

a) For \( x^2 + 7x + 12 \), \( b = 7 \) and \( c = 12 \). Use a table to find two integers whose product is 12 and whose sum is 7. In order to have a positive product and a positive sum, both numbers must be positive.

<table>
<thead>
<tr>
<th>Factors of 12</th>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 12</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>2, 6</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>3, 4</td>
<td>12</td>
<td>7</td>
</tr>
</tbody>
</table>

Therefore, \( r \) is 3 and \( s \) is 4.  
\( x^2 + 7x + 12 = (x + 3)(x + 4) \)

b) For \( x^2 + 4x + 6 \), \( b = 4 \) and \( c = 6 \).

Since no two integers have a product of 6 and sum of 4, \( x^2 + 4x + 6 \) cannot be factored over the integers.

<table>
<thead>
<tr>
<th>Factors of 6</th>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 6</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2, 3</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

I need to find two positive integers whose product is 6 and whose sum is 4.
c) For \(x^2 - 29x + 28 \lt 0\), \(b = -29\) and \(c = 28\).

Therefore, \(r\) is \(-1\) and \(s\) is \(-28\).

\(x^2 - 29x + 28 = (x - 1)(x - 28)\)

d) For \(x^2 + 3x - 18 \lt 0\), \(b = 3\) and \(c = -18\).

Therefore, \(r\) is \(-3\) and \(s\) is \(6\).

\(x^2 + 3x - 18 = (x + 6)(x - 3)\)

e) For \(x^2 - 4x - 21 \lt 0\), \(b = -4\) and \(c = -21\).

Therefore, \(r\) is \(3\) and \(s\) is \(-7\).

\(x^2 - 4x - 21 = (x + 3)(x - 7)\)

\[\]

### Example 2 Dimensions of a Water Garden

**a)** Determine binomials that represent the dimensions of the rectangular water garden.

**b)** Determine the dimensions if \(x\) represents 1 m.

**Solution**

**a)** Factor the quadratic expression for the area. Find two integers whose product is 6 and whose sum is 5. The integers are 2 and 3.

\[x^2 + 5x + 6 = (x + 2)(x + 3)\]

The dimensions can be represented by \(x + 2\) and \(x + 3\).
b) Substitute \( x = 1 \) into \( x + 2 \) and \( x + 3 \).
\[
1 + 2 = 3 \\
1 + 3 = 4
\]
The water garden has dimensions of 3 m by 4 m.

**Key Concepts**

- To factor a quadratic expression of the form \( x^2 + bx + c \), first find two integers whose product is \( c \) and whose sum is \( b \).
  - For \( x^2 + 12x + 27 \), find two integers whose product is 27 and whose sum is 12. The integers are 3 and 9.
  - Express the quadratic expression as a product, \( x^2 + bx + c = (x + r)(x + s) \).
  \[
x^2 + 12x + 27 = (x + 3)(x + 9)
\]
- Not all quadratic expressions of the form \( x^2 + bx + c \) can be factored over the integers.

**Communicate Your Understanding**

\[ \text{C1} \] Describe how to use algebra tiles to factor \( x^2 + 8x + 7 \).

\[ \text{C2} \] Explain the steps you would use to factor \( y^2 - 6y - 40 \).

\[ \text{C3} \] Explain why \( k^2 + 5k - 9 \) cannot be factored over the integers.

**Practise**

1. Illustrate the factoring of each trinomial using algebra tiles or a diagram.
   \[ \text{a) } x^2 + 4x + 3 \quad \text{b) } x^2 + 7x + 10 \]
   \[ \text{c) } x^2 + 6x + 8 \quad \text{d) } x^2 + 4x + 4 \]

2. Find two integers with the given product and sum.
   \[ \text{a) product = 45, sum = 14} \]
   \[ \text{b) product = 6, sum = -5} \]
   \[ \text{c) product = -10, sum = 3} \]
   \[ \text{d) product = -20, sum = -8} \]

3. Factor, if possible.
   \[ \text{a) } x^2 + 7x + 10 \quad \text{b) } j^2 + 12j + 27 \]
   \[ \text{c) } k^2 + 5k + 4 \quad \text{d) } p^2 + 9p + 12 \]
   \[ \text{e) } w^2 + 11w + 25 \quad \text{f) } d^2 + 10d + 24 \]

4. Factor, if possible.
   \[ \text{a) } m^2 - 7m + 10 \quad \text{b) } x^2 - 5x + 7 \]
   \[ \text{c) } y^2 - 5y + 4 \quad \text{d) } r^2 - 16r + 64 \]
   \[ \text{e) } w^2 - 9w + 24 \quad \text{f) } q^2 - 10q + 9 \]

For help with questions 3 to 5, see Example 1.
5. Factor, if possible.
   a) \(a^2 - 3a - 10\)  
   b) \(s^2 + 3s - 10\)  
   c) \(d^2 - 8d - 9\)  
   d) \(f^2 + 7f - 6\)  
   e) \(g^2 - 5g - 14\)  
   f) \(r^2 + 2r - 6\)  
   g) \(x^2 + x - 42\)  
   h) \(b^2 - 2b - 4\)

10. Expand each pair of binomials. Compare the answers.
   a) \((x + 1)(x + 3)\) and \((x + y)(x + 3y)\)
   b) \((x + 4)(x - 6)\) and \((x + 4y)(x - 6y)\)
   c) \((x - 2)(x + 9)\) and \((x - 2y)(x + 9y)\)
   d) \((x - 6)(x - 9)\) and \((x - 6y)(x - 9y)\)

11. Factor. How does the additional variable change your thinking?
   a) \(a^2 + 11ab + 24b^2\)
   b) \(k^2 - 11km + 18m^2\)
   c) \(c^2 + 4cd - 21d^2\)
   d) \(x^2 - 6xy - 16y^2\)

12. a) Make up an example of a quadratic expression that cannot be factored.
    b) Explain why it cannot be factored.

13. A parabola has equation \(y = x^2 - 4x - 12\).
   a) Factor the right side of the equation.
   b) Identify the x-intercepts of the parabola.
   c) Find the equation of the axis of symmetry, find the vertex, and draw the graph.

14. The height of a ball thrown from the top of a building can be approximated by the formula \(h = -5t^2 + 15t + 20\), where \(t\) is the time, in seconds, and \(h\) is the height, in metres.
   a) Write the formula in factored form. Hint: Remove the GCF first.
   b) How can you use the factors to find when the ball lands on the ground?

### Connect and Apply

For help with question 6, see Example 2.

6. Determine binomials that represent the length and width of each rectangle. Then, determine the dimensions of the rectangle if \(x\) represents 15 cm.

\[
\begin{align*}
\text{(a)} & \quad \text{Area is } x^2 + 18x + 80. \\
\text{(b)} & \quad \text{Area is } x^2 - 15x + 50. 
\end{align*}
\]

7. Factor completely by first removing the greatest common factor (GCF).
   a) \(3x^2 + 12x + 9\)
   b) \(2d^2 - 22d + 56\)
   c) \(5z^2 + 40z + 60\)
   d) \(4s^2 - 8s - 32\)
   e) \(bx^2 + 10bx - 24b\)
   f) \(x^3 + 18x^2 + 72x\)

8. Determine two values of \(b\) so that each expression can be factored.
   a) \(x^2 + bx + 12\)
   b) \(x^2 - bx + 4\)
   c) \(x^2 - bx - 8\)
   d) \(x^2 + bx - 10\)

9. Determine two values of \(c\) so that each expression can be factored.
   a) \(x^2 + 6x + c\)
   b) \(x^2 - x + c\)
   c) \(x^2 - 8x - c\)
   d) \(x^2 + 2x - c\)

10. Expand each pair of binomials. Compare the answers.
    a) \((x + 1)(x + 3)\) and \((x + y)(x + 3y)\)
    b) \((x + 4)(x - 6)\) and \((x + 4y)(x - 6y)\)
    c) \((x - 2)(x + 9)\) and \((x - 2y)(x + 9y)\)
    d) \((x - 6)(x - 9)\) and \((x - 6y)(x - 9y)\)

### Extend

15. a) Explain how \(x^4 + 9x^2 + 20\) and \(x^2 + 9x + 20\) are alike. How are they different?
    b) Factor \(x^4 + 9x^2 + 20\).

16. Refer to question 15. Factor.
    a) \(x^4 + 11x^2 + 30\)
    b) \(x^4 - 7x^2y^2 + 12y^2\)
    c) \(x^6 - 3x^3 - 54\)
    d) \(3(x - 5)^2 + 27(x - 5) - 66\)

17. Math Contest
    a) Expand \((x + 2)^3\).
    b) Factor \(x^3 + 9x^2 + 27x + 27\).
    c) Factor \(8a^3 + 60a^2b + 150ab^2 + 125b^3\).