Factor Quadratic Expressions of the Form \( ax^2 + bx + c \)

The Ontario Summer Games are held every two years in even-numbered years to provide sports competition for youth between the ages of 11 and 22. At the Games, approximately 2500 athletes from across the province compete in 19 sports.

Beach volleyball is one of the sports on the Games program. It is played by two teams of two players on a sand court with area given by \( 3x^2 + 10x + 3 \). Algebraic expressions for the dimensions of the court can be found by factoring the trinomial expression.

Investigate

How can you use a model to factor quadratic expressions of the form \( ax^2 + bx + c \)?

1. Use algebra tiles to form a rectangle to model the product \((2x + 1)(x + 2)\).

2. Arrange the algebra tiles representing the trinomial \(2x^2 + 5x + 3\) to form a rectangle. Identify the binomials that represent the length and width of the rectangle.

3. Repeat step 2 for each trinomial.
   a) \(3x^2 + 5x + 2\)          b) \(4x^2 + 8x + 3\)

4. Each trinomial represents the area of a rectangle. Draw diagrams and identify the binomials that represent the length and width of the rectangle.
   a) \(2x^2 + 5x + 2\)          b) \(5x^2 + 8x + 3\)

5. Reflect Describe how to use algebra tiles to factor a quadratic trinomial of the form \( ax^2 + bx + c \).

6. Reflect Can you see a way to factor trinomials of the form \( ax^2 + bx + c \) without using algebra tiles? If so, describe it.
When you expand two binomials, you add the two middle terms.

\[(2x + 3)(3x + 4) = 6x^2 + 8x + 9x + 12\]
\[= 6x^2 + 17x + 12\]

Notice the following patterns.

\[8 + 9 = 17\text{ and }8 \times 9 = 6 \times 12\]

You can use these patterns and the method of factoring by grouping to factor trinomials of the form \(ax^2 + bx + c\). Work in reverse by replacing the middle term with two terms whose integer coefficients have a product of \(a \times c\) and a sum of \(b\).

**Example 1  Break up the Middle Term**

Factor, if possible.

**a)** \(3x^2 + 8x + 4\)

**b)** \(3x^2 + 2x + 4\)

**c)** \(6x^3 - 5x + 1\)

**Solution**

**a)** For \(3x^2 + 8x + 4\), \(a = 3, b = 8\), and \(c = 4\).

Use a table to find two integers whose product is \(3 \times 4\), or 12, and whose sum is 8. In order to have a positive product and a positive sum, both integers must be positive.

<table>
<thead>
<tr>
<th>Factors of 12</th>
<th>Product</th>
<th>Sum</th>
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<tbody>
<tr>
<td>1, 12</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>2, 6</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>3, 4</td>
<td>12</td>
<td>7</td>
</tr>
</tbody>
</table>

Since the integers 2 and 6 satisfy this product and sum, break up 8x into \(2x + 6x\). Then, factor by grouping.

\[3x^2 + 8x + 4\]
\[= 3x^2 + 2x + 6x + 4\]
\[= (3x^2 + 2x) + (6x + 4)\]
\[= x(3x + 2) + 2(3x + 2)\]
\[= (3x + 2)(x + 2)\]
b) For $3x^2 + 2x + 4$, $a = 3$, $b = 2$, and $c = 4$.
Since there is no pair of integers that satisfy these conditions, $3x^2 + 2x + 4$ is not factorable over the integers.

c) For $6x^2 - 5x + 1$, $a = 6$, $b = -5$, and $c = 1$.
Since the integers $-2$ and $-3$ satisfy this product and sum, break up $-5x$ into $-2x - 3x$. Then, factor by grouping.

$$
6x^2 - 5x + 1
= 6x^2 - 2x - 3x + 1
= (6x^2 - 2x) + (-3x + 1)
= 2x(3x - 1) - 1(3x - 1)
= (3x - 1)(2x - 1)
$$

Example 2 Trinomials With Two Variables

Factor $10x^2 - 3xy - 4y^2$.

Solution

For $10x^2 - 3xy - 4y^2$, $a = 10$, $b = -3$, and $c = -4$.

$$10x^2 - 3xy - 4y^2
= 10x^2 + 5xy - 8xy - 4y^2
= (10x^2 + 5xy) + (-8xy - 4y^2)
= 5x(2x + y) - 4y(2x + y)
= (2x + y)(5x - 4y)$$

Break up $-3xy$ into $5xy - 8xy$.
Factor by grouping.
**Example 3  Remove a Common Factor**

Factor $16x^2 + 26x - 12$.

**Solution**

First, remove the greatest common factor (GCF), and then proceed as before.

The GCF of the polynomial $16x^2 + 26x - 12$ is 2.

To factor $8x^2 + 13x - 6$, I need to find two integers whose product is $8 \times (-6)$, or $-48$, and whose sum is 13. The integers $-3$ and $16$ work.

$16x^2 + 26x - 12 = 2(8x^2 + 13x - 6)$

$= 2(8x^2 - 3x + 16x - 6)$

$= 2[(8x^2 - 3x) + (16x - 6)]$

$= 2[x(8x - 3) + 2(8x - 3)]$

$= 2[(8x - 3)(x + 2)]$

$= 2(8x - 3)(x + 2)$

<table>
<thead>
<tr>
<th>Factors of $-48$</th>
<th>Product</th>
<th>Sum</th>
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<td>$-2, 24$</td>
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</tr>
<tr>
<td>$6, -8$</td>
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</tr>
</tbody>
</table>

**Key Concepts**

- Always look for a common factor first when factoring a trinomial.
- To factor $ax^2 + bx + c$, find two integers whose product is $a \times c$ and whose sum is $b$. Then, break up the middle term and factor by grouping.
- Not all quadratic expressions of the form $ax^2 + bx + c$ can be factored over the integers.

**Communicate Your Understanding**

**C1** When you use algebra tiles to factor a trinomial, why do you need to be able to form a rectangle with the tiles?

**C2** When factored, $2x^2 + 9x + 9$ can be written as $(2x + 3)(x + 3)$. Can it also be written as $(x + 3)(2x + 3)$? Justify your answer using words and a diagram.

**C3** Describe how you would factor $5x^2 + 18x + 9$. 

5.5 Factor Quadratic Expressions of the Form $ax^2 + bx + c$ • MHR 245
Practise

1. Use algebra tiles or a diagram to factor each trinomial.
   a) \(2x^2 + 5x + 3\)
   b) \(3x^2 + 7x + 4\)
   c) \(6x^2 + 5x + 1\)
   d) \(6x^2 + 11x + 4\)

For help with questions 2 to 4, see Example 1.

2. Factor, if possible.
   a) \(2x^2 + 7x + 5\)
   b) \(6y^2 + 19y + 8\)
   c) \(4k^2 + 15k + 9\)
   d) \(3m^2 + 10m + 8\)
   e) \(10w^2 + 15w + 3\)
   f) \(12q^2 + 17q + 6\)

3. Factor, if possible.
   a) \(4x^2 - 11x + 6\)
   b) \(5n^2 - 11n + 6\)
   c) \(6c^2 - 3c + 1\)
   d) \(6a^2 - 7a + 1\)
   e) \(9b^2 - 24b + 7\)
   f) \(15k^2 - 19k + 6\)

4. Factor, if possible.
   a) \(3y^2 + 4y - 7\)
   b) \(2m^2 + 3m - 9\)
   c) \(8k^2 - 6k - 5\)
   d) \(12y^2 + y - 1\)
   e) \(9x^2 - 15x - 4\)
   f) \(5h^2 - 14h - 3\)

For help with question 5, see Example 2.

5. Factor.
   a) \(3x^2 + 7xy + 2y^2\)
   b) \(6m^2 + 13mn + 2n^2\)
   c) \(2p^2 - 11pq + 5q^2\)
   d) \(6c^2 - 7cd - 10d^2\)
   e) \(9x^2 - 9xy - 4y^2\)
   f) \(6d^2 + de - 2e^2\)

For help with question 6, see Example 3.

6. Factor.
   a) \(8k^2 - 16k + 6\)
   b) \(9p^2 + 15p - 6\)
   c) \(6m^2 - 14m - 12\)
   d) \(10x^2 + 15x - 10\)
   e) \(10r^2 - 22r + 4\)
   f) \(8y^2 - 22y + 12\)

Connect and Apply

7. Factor. Then, substitute \(x = 2\) into both forms. Are the results the same? Explain.
   a) \(4x^2 + 12x + 5\)
   b) \(7x^2 - 23x + 6\)
   c) \(15x^2 - 2x - 8\)
   d) \(8x^2 + 14x - 4\)
   e) \(6x^2 - 19x + 15\)
   f) \(5x^2 + 18x + 9\)

8. Find two values of \(n\) so that each trinomial can be factored over the integers.
   a) \(x^2 + nx + 16\)
   b) \(3y^2 + ny + 25\)
   c) \(6a^2 + nab + 7b^2\)

9. Find two values of \(k\) so that each trinomial can be factored over the integers.
   a) \(36m^2 + 8m + k\)
   b) \(18y^2 - 42y + k\)
   c) \(kp^2 - 72pq + 16q^2\)

10. Describe the steps in determining whether you can factor \(ax^2 + bx + c\) over the integers.

11. Explain why it is easier to factor \(ax^2 + bx + c\) if \(a\) and \(c\) are prime numbers.
12. A rectangle has area defined by $6x^2 + 13x - 8$.

   a) Factor to find algebraic expressions for the length and width of the rectangle.
   b) If $x$ represents 10 cm, determine the perimeter and area of the rectangle.

13. The height, $h$, in metres, of a toy rocket at any time, $t$, in seconds, during its flight can be estimated using the formula $h = -5t^2 + 23t + 10$. Write the formula in factored form and determine when the rocket will fall to the ground.

14. Use Technology The range, $r$, in kilometres, of an airplane with full tanks at a power setting of $p$ revolutions per minute (RPM) can be modelled by the relation $r = -0.0008p^2 + 3.2p - 2400$.

   a) Use a computer algebra system to factor the trinomial.
   b) Describe the set of values that $p$ may take for this model.
   c) Determine what value of $p$ results in the maximum range.

15. The total revenue from sales of ski jackets is modelled by the expression $720 + 4x - 2x^2$. Revenue is also calculated as the product of the number of jackets sold and the price per jacket. Determine expressions for the number sold and the price per jacket. Hint: As the price increases, the number sold decreases.

16. a) Factor
   
   $$5x^2 + 11x + 2$$

   b) Write a quadratic trinomial that cannot be factored over the integers. Explain how you know.

   c) The area of a square is $81 - 72x + 16x^2$. If $x$ must be a positive integer, what is the least possible measure for the perimeter of the square?

17. Factor
   a) $5x^4 + 18x^2 + 9$
   b) $7x^4 - 13x^2y^2 + 6y^4$
   c) $6x^6 + 13x^3y^3 - 8y^6$
   d) $10m^6 - 7m^3n^2 - 12n^4$

18. Factor
   a) $2(x + a)^2 + 3(x + a) + 1$
   b) $2(x - b)^2 + 5(x - b) + 2$

19. a) A shape has area defined by $A = 8x^2 + 10x - 7$. Identify the shape(s).

   b) A solid has volume defined by $V = 4x^3 - 12x^2y + 9xy^2$. Identify the type of solid.